Chapter 8: Polar Coordinates and Vectors

8.1 Polar Coordinates

This is another way (in addition to the x-y system) of specifying the position of a point in the plane. We give the distance \( r \) of the point \( P \) from the origin \( O \) (or pole) and the angle \( \theta \) between the +x-axis (polar axis) and the segment \( OP \). See the image below for \( P = (3, 60^\circ) \) or \( (4, 210^\circ) \).

\[ \text{Image: polar coordinates diagram} \]

- Angles \( \theta \) are measured CCW from the polar axis if positive, and CW if negative.
- Angles may be measured in degrees or radians.
- If \( r = 0 \), then \( P = 0 \) for all \( \theta \); that is, \((0, \theta)\) represents the pole regardless of \( \theta \).
- If \(-r\) is negative, then \((-r, \theta)\) is the same as \((r, \theta \pm \pi)\). (Go \( r \) from pole in opposite direction.)
- \( P = (r, \theta) \) may also be represented by: \((-r, \theta \pm \pi)\) and \((r, \theta \pm 2\pi n)\), where \( n \) is any integer.

\[ \text{Image: polar to rectangular coordinates} \]

- Note the relation pictured above between polar and rectangular coordinates.
\[ r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\left( \frac{y}{x} \right) \]
\[ x = r \cdot \cos(\theta) \quad y = r \cdot \sin(\theta) \]
8.2 Graphs of Polar Equations

Graphing a polar equation using polar coordinates is done on polar graph paper. A table of values \((r, \theta)\) that satisfy the equation may be prepared, or technology may assist in sketching the graph.

\[ r = 2 \cdot \sin(4\theta) \]

•Note: On the TI-86, from the Home screen, set Mode to Pol to graph in polar coordinates. To have values when tracing expressed in polar coordinates, go Graph \rightarrow Formt \rightarrow PolarGC.

•Note: Polar graphs may be produced in Mathematica using the command (function) PolarPlot[ ].

•Note: The student should be familiar with the more common polar graphs (see text p594).

Circles and spirals are of the form \(r = a, r = a \cdot \sin(\theta), r = a \cdot \cos(\theta)\), and \(r = a \theta\)

Limacons are of the form \(r = a \pm b \cdot \sin(\theta)\) or \(r = a \pm b \cdot \cos(\theta)\)

Roses are of the form \(r = a \cdot \sin(n\theta)\) and \(r = a \cdot \cos(n\theta)\); \(n\) petals if \(n\) odd, \(2n\) petals if \(n\) even

Lemniscates (figure-8s) are of the form \(r^2 = a^2 \cdot \sin(2\theta)\) or \(r^2 = a^2 \cdot \cos(2\theta)\)

•Example: Convert rectangular coordinates \((\sqrt{8}, \sqrt{8})\) to polar coordinates.

\[ r^2 = (\sqrt{8})^2 + (\sqrt{8})^2 = 16; \text{choose positive } r = 4. \]
\[ \tan(\theta) = \sqrt{8} / \sqrt{8} = 1; \text{choose } \theta \text{ so the point is in quadrant I to match } (x, y) \rightarrow 45^\circ \]
Thus the point in polar coordinates is \((r, \theta) = (4, 45^\circ)\) or \((4, \pi/4)\).

•Example: Convert polar coordinates \((4, \pi/6)\) to rectangular coordinates.

\[ x = r \cdot \cos(\theta) = 4 \cdot \cos(30^\circ) = 4 \cdot \sqrt{3}/2 = 2\sqrt{3} \]
\[ y = r \cdot \sin(\theta) = 4 \cdot \sin(30^\circ) = 4 \cdot 1/2 = 2 \]
Thus the point in rectangular coordinates is \((x, y) = (2\sqrt{3}, 2)\)

•Example: Convert to polar form the equation \(y = x^2\).
Write \(r \cdot \sin(\theta) = [r \cdot \cos(\theta)]^2 = r^2 \cdot \cos^2(\theta)\) and divide by \(r\).
\[ \sin(\theta) / \cos^2(\theta) = r \text{ or } r = \tan(\theta) \cdot \sec(\theta) \]
• Example: Convert to rectangular form the equation \( r = 1 + \cos(\theta) \).

Multiply both sides by \( r \rightarrow r^2 = r + r \cdot \cos(\theta) \)

\[
(x^2 + y^2) = \sqrt{(x^2 + y^2) + x} \text{ or squaring } \rightarrow [(x^2 + y^2) - x]^2 = x^2 + y^2
\]
8.3 Polar Form of Complex Numbers; DeMoivre's Theorem

One may graph complex numbers in the complex plane. This plane has the real number line as x-axis and the imaginary number line as y-axis. The complex number $z = a + b \cdot i$ is represented as the point $(a, b)$. Images below are from Wikipedia.

- We define the modulus (or absolute value) of the complex number $z = a + b \cdot i$ as the distance of $(a, b)$ from the origin: $|z| = \sqrt{a^2 + b^2}$

- We may write the complex number $z = a + b \cdot i$ in polar or trigonometric form:
  
  $z = r \cdot (\cos(\theta) + i \cdot \sin(\theta))$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$

  $r$ is the modulus of $z$ and $\theta$ is an argument of $z$.

- Note $\text{Re}|z| = a = r \cdot \cos(\theta)$ and $\text{Im}|z| = b = r \cdot \sin(\theta)$

- Note we often write $z = r \cdot (\cos(\theta) + i \cdot \sin(\theta))$ as $z = r \cdot \text{cis}(\theta)$

- We define the complex conjugate of $z = a + b \cdot i$ as $z^* = a - b \cdot i$ (sometimes written $z^*$).

  Note in trigonometric form $\bar{z} = r \cdot (\cos(-\theta) + i \cdot \sin(-\theta)) = r \cdot (\cos(\theta) - i \cdot \sin(\theta))$

Multiplication, division, powers, and roots of complex numbers are often more easily done when the numbers are in trigonometric form.

- Multiplication: Given $z_1 = r_1 \cdot (\cos(\theta_1) + i \cdot \sin(\theta_1))$ and $z_2 = r_2 \cdot (\cos(\theta_2) + i \cdot \sin(\theta_2))$, then $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \left[\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2)\right]$

  (To multiply a real number $c$ times a complex number, just multiply the modulus times $c$.)

- Division: Given $z_1 = r_1 \cdot (\cos(\theta_1) + i \cdot \sin(\theta_1))$ and $z_2 = r_2 \cdot (\cos(\theta_2) + i \cdot \sin(\theta_2))$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos(\theta_1 - \theta_2) + i \cdot \sin(\theta_1 - \theta_2)\right]$
• Powers (De Moivre's Theorem): If \( z = r \left( \cos(\theta) + i \sin(\theta) \right) \), then for any integer \( n \):
\[
z^n = r^n \left[ \cos(n\theta) + i \sin(n\theta) \right]
\]

• Roots: If \( z = r \left( \cos(\theta) + i \sin(\theta) \right) \), and \( n \) is a positive integer, then
\[
\sqrt[n]{z} = \sqrt[n]{r} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right] \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1.
\]

• Example: Convert \( z = 2\sqrt{3} + 2i \) to trigonometric form.
\[
r = |z| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4 \quad \text{and} \quad \theta = \tan^{-1} \left( \frac{2}{2\sqrt{3}} \right) = 30^\circ
\]
Thus \( z = 4 \left[ \cos(30^\circ) + i \sin(30^\circ) \right] = 4 \cdot \text{cis}(30^\circ) = 4 \cdot \cos \left( \frac{\pi}{6} \right) + i \cdot \sin \left( \frac{\pi}{6} \right) \)

• Example: Given \( z = 2\sqrt{3} + 2i \), find \( z^3 \) and \( \sqrt[3]{z} \).
In trigonometric form, \( z = 4 \left[ \cos(30^\circ) + i \sin(30^\circ) \right] = 4 \cdot \text{cis}(30^\circ) = 4 \cdot \cos \left( \frac{\pi}{6} \right) + i \cdot \sin \left( \frac{\pi}{6} \right) \)
\[
z^3 = 4^3 \left[ \cos(3 \cdot 30^\circ) + i \sin(3 \cdot 30^\circ) \right] = 64 \left[ \cos(90^\circ) + i \sin(90^\circ) \right] = 64 \cdot \text{cis}(90^\circ) = 0 + 64i
\]
\[
\sqrt[3]{z} = \sqrt[3]{4} \left[ \cos \left( \frac{30^\circ + 360^\circ \cdot k}{3} \right) + i \sin \left( \frac{30^\circ + 360^\circ \cdot k}{3} \right) \right]
\]
\[
= \sqrt[3]{4} \cdot \text{cis}(10^\circ), \sqrt[3]{4} \cdot \text{cis}(130^\circ), \sqrt[3]{4} \cdot \text{cis}(250^\circ)
\]

• Note in complex numbers, there are 3 cube roots, 4 fourth roots, and so on. If we graph them, they are equally spaced around a circle of radius \( |z| \).

• On the TI-86, represent complex numbers as \((a, b)\). Most operations are available from the Home screen. The CPLX menu aids in converting between polar and rectangular, and offers to find moduli, arguments, and more.

• Example: Given \( z_1 = \sqrt{3} + i \), and \( z_2 = 1 + \sqrt{3}i \), find \( z_1 \cdot z_2 \) and \( \frac{z_1}{z_2} \).
\[
z_1 = 2 \cdot \text{cis}(30^\circ) \quad \text{and} \quad z_2 = 2 \cdot \text{cis}(60^\circ)
\]
Thus:
\[
z_1 \cdot z_2 = 2 \cdot 2 \cdot \text{cis}(30^\circ + 60^\circ) = 4 \cdot \text{cis}(90^\circ) = 0 + 4i
\]
\[
\frac{z_1}{z_2} = \frac{2}{2} \cdot \text{cis}[30^\circ - 60^\circ] = 1 \cdot \text{cis}(-30^\circ) = 1 \left[ \cos(30^\circ) - i \sin(30^\circ) \right] = \frac{\sqrt{3}}{2} - \frac{1}{2}i
\]

Mathematica handles complex numbers entered as \(a + bi\), where the \(i\) is a special symbol.
8.4 Vectors

A vector is a mathematical object with magnitude and direction. Geometrically, it is represented by a directed line segment or arrow. (Images below from Wikipedia)

- For a vector from initial point \(A(x_1, y_1)\) to terminal point \(B(x_2, y_2)\), we write:
  \[
  \overrightarrow{AB} = <x_2 - x_1, y_2 - y_1>.
  \]
  For a vector starting at the origin, \(\overrightarrow{OA} = <2, 3> = 2\hat{i} + 3\hat{j}\)
  The notation \(\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}\) describes a vector with components along the x, y, and z axes of a, b, and c, where \(\hat{i}, \hat{j}, \hat{k}\) are unit vectors (of length one) pointing along the coordinate axes.

  Sometimes vectors are represented in print by boldface quantities. The TI-86 calculator inputs vectors in square brackets. Thus \(\mathbf{u} = [2, 3]\) is sometimes seen.

- For a vector from initial point \(A(x_1, y_1)\) to terminal point \(B(x_2, y_2)\), define the length or magnitude of the vector
  \[
  |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
  \]
  and its direction, measured CW from the +x-axis, as
  \[
  \theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)
  \]
  - The zero vector is defined to have length zero.

- Geometrically, multiplication of a vector by a scalar multiplies the length of the vector of its arrow representation. If \(\mathbf{a} = <x_1, y_1>\), \(c \cdot \mathbf{a} = <c \cdot x_1, c \cdot y_1>\).
  The negative of a vector has length unchanged but points in the opposite direction.
  If \(\mathbf{a} = <x_1, y_1>\), \(-\mathbf{a} = <-x_1, -y_1>\).

- Addition: If the vectors \(\mathbf{a} = <x_1, y_1>\) and \(\mathbf{b} = <x_2, y_2>\) are tail to tail, form the parallelogram with \(\mathbf{a}\) and \(\mathbf{b}\) as two consecutive sides. The diagonal starting at the initial point of the vectors is the sum \(\mathbf{a} + \mathbf{b}\). Alternatively, move \(\mathbf{b}\) parallel to its own length so its tail is at the tip of \(\mathbf{a}\). The vector from the tail of \(\mathbf{a}\) to the tip of \(\mathbf{b}\) is the sum.

  Analytically, \(\mathbf{a} + \mathbf{b} = <x_1 + x_2, y_1 + y_2>\)

- Subtraction: With \(\mathbf{a}\) and \(\mathbf{b}\) tail to tail, draw the vector from the tip of \(\mathbf{b}\) to the tip of \(\mathbf{a}\). This is equivalent to adding \(\mathbf{a} + (-\mathbf{b})\).

  Analytically, \(\mathbf{a} - \mathbf{b} = <x_1 - x_2, y_1 - y_2>\)
A unit vector in the direction of vector \( \mathbf{a} \), where \( \mathbf{a} = <x_1, y_1> \), is:

\[
\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{x_1 \hat{i} + y_1 \hat{j}}{\sqrt{x_1^2 + y_1^2}} = \left( \frac{x_1}{\sqrt{x_1^2 + y_1^2}}, \frac{y_1}{\sqrt{x_1^2 + y_1^2}} \right)
\]

The horizontal and vertical components of a vector with length \( |\mathbf{a}| \) and direction \( \theta \):

\[ x_1 = |\mathbf{a}| \cdot \cos(\theta) \quad y_1 = |\mathbf{a}| \cdot \sin(\theta) \]

Example: Express the vector from P(3, 2) to Q(8, 9) in component form.

\[ \mathbf{PQ} = (8 - 3)\hat{i} + (9 - 2)\hat{j} = 5\hat{i} + 7\hat{j} = <5, 7> \]

Example: Find \( 2\mathbf{u} \), \( -3\mathbf{v} \), \( \mathbf{u} + \mathbf{v} \), \( 3\mathbf{u} - 4\mathbf{v} \), and \( |\mathbf{u}| \) for \( \mathbf{u} = <2, 7> \) and \( \mathbf{v} = <3, 1> \)

\[ 2\mathbf{u} = <4, 14> \quad -3\mathbf{v} = <-9, -3> \quad \mathbf{u} + \mathbf{v} = <5, 8> \quad 3\mathbf{u} - 4\mathbf{v} = <-6, 17> \quad |\mathbf{u}| = \sqrt{53} \]

Example: Find the magnitude and direction of the vector \( \mathbf{v} = <3, 4> \).

\[ |\mathbf{v}| = \sqrt{3^2 + 4^2} = 5 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ \]

Example: Find the horizontal and vertical components of the vector \( \mathbf{v} \) with length 40 and direction 30\(^\circ\).

\[ \tilde{\mathbf{v}} = \left( 40 \cdot \cos(30^\circ)\hat{i} + 40 \cdot \sin(30^\circ)\hat{j} \right) = 20\sqrt{3}\hat{i} + 20\hat{j} \]
8.5 Dot Product

• If $\mathbf{A} = <x_1, y_1>$ and $\mathbf{B} = <x_2, y_2>$ are vectors as pictured above, then define:

$$\mathbf{A} \cdot \mathbf{B} = x_1x_2 + y_1y_2 = |\mathbf{A}||\mathbf{B}||\cos(\theta) = \text{dot (or scalar or inner) product of } \mathbf{A} \text{ and } \mathbf{B}.$$  

Note this product is a scalar, not a vector. It has magnitude without direction.

• Example: Given $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 4\mathbf{i} - 3\mathbf{j}$. Find $\mathbf{A} \cdot \mathbf{B}$ and the angle $\theta$ between the vectors

$$\mathbf{A} \cdot \mathbf{B} = (3)(4) + (4)(-3) = 0 \text{ and } \cos(\theta) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{0}{(5)(5)} = 0 \Rightarrow \theta = 90^\circ$$

• If two nonzero vectors have dot product $= 0$, then the vectors are perpendicular.

(Converse also true)

• The angle between two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ is given by:

$$\cos(\theta) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

• Note for vector $\mathbf{u}$: $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

• The scalar projection of $\mathbf{A}$ onto $\mathbf{B}$ (also called the component of $\mathbf{A}$ along or in the direction of $\mathbf{B}$)

$$= |\mathbf{A}| \cos(\theta) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2},$$  

where $\theta$ is the angle between $\mathbf{A}$ and $\mathbf{B}$. (See diagram above)

• The vector projection of $\mathbf{A}$ onto $\mathbf{B}$ is the above scalar projection times a unit vector in the direction of $\mathbf{B}$.

$$= \text{proj}_B \mathbf{A} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2} \right) \mathbf{B}$$

• If the vector $\mathbf{A}$ is resolved into components $\mathbf{A}_1$ and $\mathbf{A}_2$ where $\mathbf{A}_1$ is parallel to $\mathbf{B}$ and $\mathbf{A}_2$ is perpendicular to $\mathbf{B}$, then $\mathbf{A}_1 = \text{proj}_B \mathbf{A}$ and $\mathbf{A}_2 = \mathbf{A} - \text{proj}_B \mathbf{A}$
• Example: A lawn mower is pushed a distance of 65.0 m along a horizontal path by a constant force along the handle of 200. N. The handle makes an angle of 30° with the horizontal. Find the work done by the force acting along the handle.

Force and displacement are both vectors. Work is the scalar product $F \cdot d = |F| |d| \cos(\theta)$

Thus $\text{Work} = (200 \text{ N})(65 \text{ m})(\cos30°) = 11258 \text{ Nm} \approx 11300 \text{ J}$ to 3 significant figures.

• Example: Given $\mathbf{u} = <-2, 4>$ and $\mathbf{v} = <1, 1>$. Find the component of $\mathbf{u}$ along $\mathbf{v}$, the vector projection of $\mathbf{u}$ onto $\mathbf{v}$, and resolve $\mathbf{u}$ into components $\mathbf{u}_1$ parallel to $\mathbf{v}$ and $\mathbf{u}_2$ perpendicular to $\mathbf{v}$.

Component of $\mathbf{u}$ along $\mathbf{v} = u \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{(-2)(1) + (4)(1)}{\sqrt{1^2 + 1^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\text{proj}_v \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} = \left( \sqrt{2} \right) \frac{<1,1>}{\sqrt{2}} = <1,1>$

$\mathbf{u}_1 = \text{proj}_v \mathbf{u} = <1,1>$

$\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = <-2,4> - <1,1> = <-3,3>$

• Example: A jet is flying through wind blowing at 55 mph in direction N 30° E. The jet has speed of 765 mph in still air, and is heading in the direction N 45° E. Find the resultant speed and direction of the jet.

Write both velocity vectors in component form. Add, and then convert back to magnitude and direction form. Wind $\mathbf{w} = <55 \cdot \cos(60°), 55 \cdot \sin(60°)> = \left( \frac{55}{2}, \frac{55\sqrt{3}}{2} \right)$

Jet $\mathbf{j} = <765 \cdot \cos(45°), 765 \cdot \sin(45°)> = \left( \frac{765\sqrt{2}}{2}, \frac{765\sqrt{2}}{2} \right)$

$\mathbf{j} + \mathbf{w} = <568.4, 588.6>$ so $|\mathbf{j} + \mathbf{w}| = 818 \text{ mph}$ in direction $\theta = 46°$ or N 44° E